

On the characterisation of non-degenerate foliations of pseudo-Riemannian manifolds with conformally flat leaves.

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Abstract

A necessary and sufficient condition for the leaves of a *non-degenerate* foliation of a pseudo-Riemannian manifold to be conformally flat is developed. The condition mimics the classical condition of the vanishing of the Weyl or Cotton tensor establishing the conformal flatness of a pseudo-Riemannian manifold in the sense that it is also formulated in terms of the vanishing of certain tensors. These tensors play the role of the Weyl or the Cotton tensors and they are defined in terms of the distribution generating the leaves of the foliation and the curvature of a linear torsion-free connection (the *bi-conformal connection*) which is also defined from the same distribution.

1 Introduction

A classical result in differential geometry states that a pseudo-Riemannian manifold is conformally flat if and only if the Weyl tensor vanishes (in an open set) when the manifold dimension is higher than 3. If the dimension is 3 then the Weyl tensor vanishes for any pseudo-Riemannian metric, be it conformally flat or not and one must resort to the *Cotton tensor* to obtain a similar criterion as for the higher dimensional one. Both the Weyl and the Cotton tensors are defined exclusively in terms of the pseudo-Riemannian metric and hence they enable us to test algorithmically whether a given pseudo-Riemannian manifold is conformally flat or not.

One would like to have similar criteria to the ones just pointed out if we work with a foliation of a pseudo-Riemannian manifold. The question here is whether, given such a foliation, we can answer in an algorithmically way if the induced metric on the leaves of the foliation is conformally flat. Of course it is clear that we can address this problem by an explicit computation of the metric induced on each leaf and then using the classical results explained in the previous paragraph. However, it may happen that the foliation is given in such a way that an explicit computation of the induced metric is very difficult or impossible. For example the foliation can be given as an *involutive distribution* or equivalently as a set of *integrable* differential forms. In this case to check whether the leaves of the foliation are conformally flat by the application of the classical result cannot be done in a direct way.

In this paper we provide an algorithmic criterion to check whether a foliation has conformally flat leaves which only requires the knowledge of the distribution generating the foliation. We are able to construct a tensor which vanishes if and only if the leaves of the given foliation are conformally flat. Our criterion is thus similar to the classical criterion of the Weyl (Cotton) tensor vanishing to characterize conformally flat pseudo-Riemannian manifolds. The difference is that our tensor is constructed in terms a pair of orthogonal and complementary *projection morphisms* rather than the pseudo-Riemannian metric. These projection morphisms are defined in such a way that the range of one of the projections coincide with the vector space spanned by the distribution at each point of the manifold. This ensures that only the knowledge of the distribution is needed to apply our criterion. The only current limitation of our criterion is that it is only valid for a *non-degenerate* foliation (the induced metric on the leaves is non-degenerate).

The study of foliations of pseudo-Riemannian manifolds with conformally flat leaves has relevance in certain applications. For example if one knows in advance that a foliation with conformally flat leaves

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exists one may be able to find adapted coordinated systems exploiting this. Related to this is the existence problem of this type of foliations. This existence problem has been addressed in very particular cases and with very specific motivations [2, 7]. The results presented in this paper enables us to address the existence problem in full generality.

This paper is structured as follows: in section 2 we present some generalities about non-degenerate smooth distributions which are essential for this work. Section 3 introduces the *bi-conformal connection* which is a linear torsionless connection which is defined in any pseudo-Riemannian manifold possessing a non-degenerate distribution. The main result of the paper is Theorem 1 of section 4 where we present a necessary and sufficient condition guaranteeing that the leaves of a foliation are conformally flat. As stated above this condition takes the form of the vanishing of a certain tensor which play the role of either the Weyl or the Cotton tensor. This tensor is indeed defined by means of the curvature of the bi-conformal connection and it is of rank four or three depending on the dimension of the leaves. Specific applications of Theorem 1 are presented in section 5. All the computations in this paper were carried out with the *Mathematica* suite *xAct* (see [5, 6]).

2 Non-degenerate smooth distributions on pseudo-Riemannian manifolds

Let (V, g) be a N -dimensional smooth pseudo-Riemannian manifold and denote by ∇ its torsion-free Levi-Civita connection. In this work we will mostly use abstract indices but sometimes index-free notation will be employed. In this case boldface symbols will be used to denote tensorial quantities. Abstract indices of tensors and tensor fields arising from the tensor bundle of V will be denoted with small Latin letters. Indices enclosed in round (square) brackets denote the operation of symmetrization (antisymmetrization). The Levi-Civita connection ∇ defines the Riemann and Ricci tensors in the standard way. Our conventions to define these quantities are laid by the relations

$$\nabla_a \nabla_b X^c - \nabla_b \nabla_a X^c = R_{bad}{}^c X^d, \quad R_{ac} = R_{abc}{}^b.$$

Indices are lowered and raised with the metric g_{ab} and its inverse g^{ab} (in index-free notation this is represented by g^\sharp). Assume now that we have defined a *non-degenerate smooth distribution* on the manifold V . Recall that a distribution is a smooth map $D : V \rightarrow T(V)$ such that for any point $x \in V$ its image D_x under D is a vector subspace of $T_x(V)$. The smoothness ensures that the dimension of D_x does not depend on the point x and shall be called the dimension of the distribution D . The distribution D is non-degenerate if the bi-linear form $g|_x$ restricted to the vector space D_x is non-degenerate. In this case we can decompose the vector space $T_x(V)$ in the form

$$T_x(V) = D_x \oplus D_x^\perp,$$

where D_x^\perp is the orthogonal complement of D_x with respect to the scalar product defined by g_x . The vector spaces D_x, D_x^\perp can be respectively characterized by means of a pair of orthogonal projectors $\mathbf{P}_x, \mathbf{\Pi}_x$. These are endomorphisms of $T_x(V)$ (projection morphisms) and one has

$$D_x = \mathbf{P}_x(T_x(V)), \quad D_x^\perp = \mathbf{\Pi}_x(T_x(V)), \quad \mathbf{\Pi}_x + \mathbf{P}_x = \mathbb{1}, \quad \mathbf{\Pi}_x \circ \mathbf{P}_x = \mathbf{P}_x \circ \mathbf{\Pi}_x = 0, \quad (1)$$

$$\mathbf{\Pi}_x \circ \mathbf{\Pi}_x = \mathbf{\Pi}_x, \quad \mathbf{P}_x \circ \mathbf{P}_x = \mathbf{P}_x \quad (2)$$

Since the distribution D is smooth we can define smooth sections $\mathbf{P}, \mathbf{\Pi}$ on the bundle of 1-1 tensors $T_1^1(V)$ with $\mathbf{P}|_x = \mathbf{P}_x$ and $\mathbf{\Pi}|_x = \mathbf{\Pi}_x, x \in V$. If we represent these sections in index notation then is clear from (1)-(2) that they must have the properties

$$P_{ab}P_c{}^b = P_{ac}, \quad \Pi_{ab}\Pi_c{}^b = \Pi_{ac}, \quad P_{ab}\Pi_c{}^b = 0, \quad P_{ab} + \Pi_{ab} = g_{ab}. \quad (3)$$

Reciprocally, we can start by defining smooth sections fulfilling the relations given by the previous set of equations. These sections define smooth distributions D, D^\perp by means of the definitions

$$D_x = \mathbf{P}|_x(T_x(V)), \quad D_x^\perp = \mathbf{\Pi}|_x(T_x(V)).$$

Therefore we can speak either of a non-degenerate distribution or the smooth sections \mathbf{P} and $\mathbf{\Pi}$ indifferently. Indeed we can just work with one projector as the other one is completely determined by the

relation $\mathbf{\Pi} + \mathbf{P} = \mathbb{1}$. We will use the symbols $D(\mathbf{P})$, $D(\mathbf{\Pi})$ to denote the smooth distributions generated respectively by \mathbf{P} , $\mathbf{\Pi}$. The dimensions of these distributions are given by

$$\dim(D(\mathbf{P})) = \text{tr}(\mathbf{P}) = P^a_a, \quad \dim(D(\mathbf{\Pi})) = \text{tr}(\mathbf{\Pi}) = \Pi^a_a.$$

For any distribution D we define the set $\mathfrak{X}(D)$ by

$$\mathfrak{X}(D) \equiv \{\vec{\xi} \in \mathfrak{X}(V) : \vec{\xi}|_x \in D_x, \forall x \in V\}, \quad (4)$$

where $\mathfrak{X}(V)$ is the set of smooth vector fields on V . A distribution is said to be involutive if for any $\vec{\xi}_1, \vec{\xi}_2$ in $\mathfrak{X}(D)$, the Lie bracket $[\vec{\xi}_1, \vec{\xi}_2]$ is also in $\mathfrak{X}(D)$.

3 The Bi-conformal connection

For any pair P_{ab} , Π_{ab} fulfilling the conditions of (3), define the following tensors

$$M_{abc} \equiv \nabla_b P_{ac} + \nabla_c P_{ab} - \nabla_a P_{bc}, \quad E_a \equiv M_{acb} P^{cb}, \quad W_a \equiv -M_{acb} \Pi^{cb}. \quad (5)$$

In terms of these quantities we define the following object

$$L^a_{bc} \equiv \frac{1}{2\text{tr}(\mathbf{P})} (E_b P^a_c + E_c P^a_b) + \frac{1}{2(N - \text{tr}(\mathbf{P}))} (W_b \Pi^a_c + W_c \Pi^a_b) + \frac{1}{2} (P^a_c - \Pi^a_c) M^c_{bc}. \quad (6)$$

This is a (1,2)-tensor with the symmetries $L^a_{bc} = L^a_{cb}$. In [3] this tensor was used to introduce a new symmetric connection in the manifold (V, g) . We recall now this definition.

Definition 3.1. *Let (V, g) be a pseudo-Riemannian manifold with its standard Levi-Civita connection ∇ and let \mathbf{P} , $\mathbf{\Pi}$ be any pair fulfilling (3). Use them to define the quantity L^a_{bc} by means of (6). The bi-conformal connection is by definition the only linear connection such that its difference with the Levi-Civita connection results in the tensor L^a_{bc} .*

Recall that the difference between two linear connections is always a “true tensor”, even though the connections themselves are not. If we denote the components of the Bi-conformal connection in a given frame by $\bar{\gamma}^a_{bc}$ then the previous definition means that

$$\bar{\gamma}^a_{bc} \equiv \gamma^a_{bc} + L^a_{bc}, \quad (7)$$

where γ^a_{bc} are the components of the Levi-Civita connection in the same frame (here and in the following, we write frame-indices adopting explicit numerical values in bold).

We denote the covariant derivative associated to the bi-conformal connection by $\bar{\nabla}$. Given that L^a_{bc} is symmetric in its last two indices we may conclude that the bi-conformal connection has no torsion. The Riemann and Ricci tensors of the bi conformal connection are denoted by

$$\bar{R}_{abc}{}^d, \quad \bar{R}_{ab} \equiv \bar{R}_{acb}{}^c$$

It is possible to check in explicit examples that $\bar{R}_{abc}{}^c \neq 0$, hence the bi-conformal connection is not a metric connection in general which means that \bar{R}_{ab} need not be symmetric.

To gain some geometric intuition about the bi-conformal connection it is instructive to compute its components in a frame adapted to the distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$. Let $\{\vec{e}_1, \dots, \vec{e}_N\}$ be such a frame. We adopt the following conventions

$$D(\mathbf{P}) = \text{Span}\{\vec{e}_1, \dots, \vec{e}_p\}, \quad D(\mathbf{\Pi}) = \text{Span}\{\vec{e}_{1+p}, \dots, \vec{e}_N\}, \quad (8)$$

where p is the dimension of the distribution $D(\mathbf{P})$ (hence $p = \text{tr}(\mathbf{P})$). Since the frame is adapted to the distribution, we have the relations

$$P_{AB} = g(\vec{e}_A, \vec{e}_B), \quad \Pi_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta), \quad P_{\alpha\beta} = \Pi_{AB} = P_{\alpha B} = P_{A\beta} = \Pi_{\alpha B} = \Pi_{A\beta} = 0, \quad (9)$$

where we have adopted the convention that capital boldface Latin letters represent frame indices with numerical values in the range $(1, p)$ and Greek bold symbols are frame indices in the range $(p + 1, N)$.

Our convention for the definition of the components of Levi-Civita connection (and any other linear connection) in a frame is

$$\nabla_{\vec{e}_a} \vec{e}_b = \gamma^c_{ab} \vec{e}_c. \quad (10)$$

The Lie brackets of the elements of this frame are characterized by the structure functions C^c_{ab} defined by

$$[\vec{e}_a, \vec{e}_b] = C^c_{ab} \vec{e}_c = \nabla_{\vec{e}_a} \vec{e}_b - \nabla_{\vec{e}_b} \vec{e}_a, \quad (11)$$

where the last equation holds because the Levi-Civita connection has no torsion (indeed it holds for any torsion-free connection). To compute the components of the bi-conformal connection in this frame, we just need to compute the components of the tensor L^a_{bc} and then use (7). In order to do this we are required to compute the components of P_{ab} , Π_{ab} , $\nabla_a P_{bc}$ and $\nabla_a \Pi_{bc}$ and then use eqs (5)-(6). This is achieved using the standard definition of covariant derivative of a tensor

$$(\nabla_{\vec{e}_a} \mathbf{P})(\vec{e}_b, \vec{e}_c) = \vec{e}_a(\mathbf{P}(\vec{e}_b, \vec{e}_c)) - \mathbf{P}(\nabla_{\vec{e}_a} \vec{e}_b, \vec{e}_c) - \mathbf{P}(\vec{e}_b, \nabla_{\vec{e}_a} \vec{e}_c), \quad (12)$$

$$(\nabla_{\vec{e}_a} \mathbf{\Pi})(\vec{e}_b, \vec{e}_c) = \vec{e}_a(\mathbf{\Pi}(\vec{e}_b, \vec{e}_c)) - \mathbf{\Pi}(\nabla_{\vec{e}_a} \vec{e}_b, \vec{e}_c) - \mathbf{\Pi}(\vec{e}_b, \nabla_{\vec{e}_a} \vec{e}_c), \quad (13)$$

and replacing the covariant derivatives of the right hand sides by means of (10). The differentials $\vec{e}_a(\mathbf{P}(\vec{e}_b, \vec{e}_c))$, $\vec{e}_a(\mathbf{\Pi}(\vec{e}_b, \vec{e}_c))$ are replaced using the condition of ∇ being the Levi-Civita connection

$$\vec{e}_a(g(\vec{e}_b, \vec{e}_c)) = g(\nabla_{\vec{e}_a} \vec{e}_b, \vec{e}_c) + g(\vec{e}_b, \nabla_{\vec{e}_a} \vec{e}_c), \quad (14)$$

which via (9) yields

$$\vec{e}_A(\mathbf{P}(\vec{e}_B, \vec{e}_C)) = g(\nabla_{\vec{e}_A} \vec{e}_B, \vec{e}_C) + g(\vec{e}_B, \nabla_{\vec{e}_A} \vec{e}_C), \quad (15)$$

$$\vec{e}_\alpha(\mathbf{\Pi}(\vec{e}_\beta, \vec{e}_\gamma)) = g(\nabla_{\vec{e}_\alpha} \vec{e}_\beta, \vec{e}_\gamma) + g(\vec{e}_\beta, \nabla_{\vec{e}_\alpha} \vec{e}_\gamma). \quad (16)$$

From these formulae we deduce that the components of L^a_{bc} in the adapted frame will only be given in terms of the components of the Levi-Civita connection in the same frame and the something similar will happen with the components of the bi-conformal connection. The explicit relations are obtained after performing a certain amount of elementary algebra and they are

$$\bar{\gamma}^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma}, \quad \bar{\gamma}^A_{BC} = \gamma^A_{BC}, \quad (17a)$$

$$\bar{\gamma}^A_{B\alpha} = \text{TF}(\gamma^A_{B\alpha}), \quad \bar{\gamma}^A_{\alpha B} = \gamma^A_{\alpha B} - \gamma^A_{B\alpha} + \text{TF}(\gamma^A_{B\alpha}), \quad (17b)$$

$$\bar{\gamma}^\alpha_{\beta A} = \text{TF}(\gamma^\alpha_{\beta A}), \quad \bar{\gamma}^\alpha_{A\beta} = \gamma^\alpha_{A\beta} - \gamma^\alpha_{\beta A} + \text{TF}(\gamma^\alpha_{\beta A}), \quad (17c)$$

$$\bar{\gamma}^A_{\alpha\beta} = \frac{1}{2} (\gamma^A_{\alpha\beta} - \gamma^A_{\beta\alpha}), \quad \bar{\gamma}^\alpha_{AB} = \frac{1}{2} (\gamma^\alpha_{AB} - \gamma^\alpha_{BA}), \quad (17d)$$

where (we follow Einstein's summation convention for frame indices)

$$\text{TF}(\gamma^A_{B\gamma}) \equiv \frac{1}{2} (\gamma^A_{B\gamma} + \gamma^D_{C\gamma} P^{AC} P_{DB}) - \frac{1}{p} \gamma^C_{C\gamma} \delta_B^A, \quad (18)$$

$$\text{TF}(\gamma^\alpha_{\beta C}) \equiv \frac{1}{2} (\gamma^\alpha_{\beta C} + \gamma^\mu_{\gamma C} P^{\alpha\gamma} P_{\mu\beta}) - \frac{1}{N-p} \gamma^\gamma_{\gamma C} \delta_\beta^\alpha, \quad (19)$$

By looking at these formulae we see that the components of the bi-conformal connection restricted to each of the distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$ are just the components of the Levi-Civita connection (see eq. (17a)). This is similar to what happens in the case of an *adapted linear connection* to the pair of distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$ (see [1] for the definition of this notion). However, we can also check by means of these formulae that the bi-conformal connection is not an adapted linear connection in general. For example, formula (17d) tells us that the quantities $\bar{\gamma}^A_{\alpha\beta}$ and $\bar{\gamma}^\alpha_{AB}$ are different from zero in any adapted frame if the distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$ are not *involutive* (see eq. (11)) whereas in the case of an adapted linear connection these quantities vanish in a certain adapted frame regardless of the properties of the distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$.

The relevance of the bi-conformal connection was established in [3, 4] where it enabled to obtain a local invariant characterization of the family of *conformally separable* pseudo-Riemannian manifolds. In the next section we are going to show another geometric feature of the bi-conformal connection dealing with non-degenerate foliations with conformally flat leaves.

4 Existence of foliations with conformally flat leaves

A smooth foliation of the manifold (V, \mathbf{g}) is a family $\{\Sigma_\tau\}$, $\tau \in I \subset \mathbb{R}^p$ of smooth embedded sub-manifolds of dimension $N - p$ such that for any $\tau \neq \tau'$, $\Sigma_\tau \cap \Sigma_{\tau'} = \emptyset$ and $V = \cup_{\tau \in I} \Sigma_\tau$. Each of the submanifolds Σ_τ is called a leaf of the foliation and the natural number p is their co-dimension. The foliation is *non-degenerate* if the induced metric (first fundamental form) on each leaf is non-degenerate everywhere. Note that any non-degenerate foliation always generates a non-degenerate distribution defined for any $x \in V$ by

$$x \mapsto T_x(\Sigma_\tau), x \in \Sigma_\tau$$

where $T_x(\Sigma_\tau)$ is the subspace of the tangent space $T_x(V)$ corresponding to the tangent space of the leaves. By Frobenius theorem, the distribution so defined is involutive. Also, associated to any foliation there is a smooth map $t : V \rightarrow \mathbb{R}^p$ defined for any $x \in V$ by $t(x) = \tau$, where $x \in \Sigma_\tau$. This function will be referred to as the level set map of the foliation.

As explained in section 2, any non-degenerate smooth distribution is characterized by a smooth projector \mathbf{P} and its orthogonal complement $\mathbf{\Pi}$, and therefore we can apply these ideas to the present situation. We choose the convention that the distribution generated by $\{\Sigma_\tau\}$ is $D(\mathbf{\Pi})$ which means that $\dim(D(\mathbf{\Pi})) = N - p$ and $\text{tr}(\mathbf{\Pi}) = N - p$, $\text{tr}(\mathbf{P}) = p$

In this work we are interested in finding necessary and sufficient conditions which guarantee that the leaves of a non-degenerate foliation are conformally flat. The following theorem supplies these conditions.

Theorem 1. *The leaves with dimension $N - p > 3$, $N > 3$ of a non-degenerate foliation of a pseudo-Riemannian manifold (V, \mathbf{g}) are conformally flat if and only if the pair of complementary projectors $(\mathbf{\Pi}, \mathbf{P})$ associated to the distribution defined by the foliation fulfill the following condition*

$$\mathbf{T}^{\parallel}(\mathbf{\Pi}) = 0, \quad (20)$$

where

$$(T^{\parallel}(\mathbf{\Pi}))_{cab}{}^d \equiv \Pi^d{}_r \Pi^s{}_c \Pi^t{}_a \Pi^q{}_b T_{stq}{}^r, \quad (21)$$

$$T_{bac}{}^d \equiv 2\bar{R}_{bac}{}^d - \frac{2}{2 - N - p}(\Pi^d{}_c L_{[ab]}^{\Pi} + \Pi^d{}_{[b} L_{a]c}^{\Pi} + \Pi_{c[a} L_{b]q}^{\Pi} \Pi^{qd}) , \quad (22)$$

and

$$L_{bc}^{\Pi} \equiv 2 \left(\Pi^d{}_r \bar{R}_{bdc}{}^r - \frac{1}{N - p} (\Pi^d{}_c \Pi^r{}_q \bar{R}_{bdr}{}^q + \Pi^d{}_b \Pi^r{}_q R_{cdr}{}^q - \Pi^r{}_q \bar{R}_{cbr}{}^q) \right) + \frac{\bar{R}^{\Pi}}{1 - N + p} \Pi_{bc}, \quad (23)$$

$$\bar{R}^{\Pi} \equiv \Pi^d{}_r \bar{R}_{bdc}{}^r \Pi^{cb}, \quad (24)$$

the bi-conformal connection in these expressions being that defined from P_{ab} and $\Pi_{ab} = g_{ab} - P_{ab}$ in the way explained in Definition 3.1. If $N - p = 3$ then eq. (20) is to be replaced by

$$\mathbf{B}^{\parallel}(\mathbf{\Pi}) = 0, \quad (25)$$

where

$$(B^{\parallel}(\mathbf{\Pi}))_{abc} \equiv \Pi_a{}^r \Pi_b{}^s \Pi_c{}^q \bar{\nabla}_{[r} L_{s]q}^{\Pi}. \quad (26)$$

Proof. We choose a frame adapted to the distributions $D(\mathbf{P})$, $D(\mathbf{\Pi})$, in the way explained in section 3, (see eq. (8)) and show that in that frame $\mathbf{T}^{\parallel}(\mathbf{\Pi}) = 0$ ($N - p > 3$) or $\mathbf{B}^{\parallel}(\mathbf{\Pi})$ ($N - p = 3$) vanish if and only if the leaves of the foliation are conformally flat. In fact, given that $D(\mathbf{\Pi})$ is involutive we can find a frame such that $\vec{e}_\alpha = \partial/\partial x^\alpha$ in certain local coordinates (x^1, \dots, x^{N-p}) . In this adapted frame the components of the induced metric on the leaves (first fundamental form) are just $\Pi_{\alpha\beta}$ and the components of the bi-conformal connection are given by (17a)-(17d). In addition, since the Levi-Civita connection has no torsion, we have the relation

$$0 = \left[\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right] = [\vec{e}_\alpha, \vec{e}_\beta] = \nabla_{\vec{e}_\alpha} \vec{e}_\beta - \nabla_{\vec{e}_\beta} \vec{e}_\alpha = (\gamma^A{}_{\alpha\beta} - \gamma^A{}_{\beta\alpha}) \vec{e}_A + (\gamma^\gamma{}_{\alpha\beta} - \gamma^\gamma{}_{\beta\alpha}) \vec{e}_\gamma, \quad (27)$$

which entails $\gamma^A_{\alpha\beta} = \gamma^A_{\beta\alpha}$. Using this in the first condition of (17d) we conclude that $\bar{\gamma}^A_{\alpha\beta} = 0$. Next we need to compute the components of the bi-conformal curvature $\bar{R}_{abc}{}^d$ in the chosen frame. To that end we use the standard definition of the curvature of a linear connection in differential geometry particularized to our frame

$$\bar{R}(\bar{e}_a, \bar{e}_b)\bar{e}_c = \bar{\nabla}_{\bar{e}_b}\bar{\nabla}_{\bar{e}_a}\bar{e}_c - \bar{\nabla}_{\bar{e}_a}\bar{\nabla}_{\bar{e}_b}\bar{e}_c - \bar{\nabla}_{[\bar{e}_a, \bar{e}_b]}\bar{e}_c. \quad (28)$$

In fact, if we look at the definition of $T^{\parallel}(\Pi)$, we realize that to compute this quantity in the adapted frame we only need to compute the scalars $\bar{R}_{\mu\beta\gamma}{}^{\lambda}$ and $L^{\Pi}_{\alpha\beta}$ which are, respectively, the only non-vanishing components of the tensors $(\bar{R}^{\parallel}(\Pi))_{cab}{}^d \equiv \Pi^d{}_r \Pi^s{}_c \Pi^t{}_a \Pi^q{}_b \bar{R}_{stq}{}^r$ and $(L^{\Pi\parallel})_{ab} \equiv \Pi^t{}_a \Pi^q{}_b L^{\Pi}_{tq}$ in the adapted frame. This is achieved by combining (28), (10) (with ∇ replaced by $\bar{\nabla}$) and (11) with the result

$$\bar{R}_{\mu\beta\gamma}{}^{\lambda} = 2\bar{e}_{[\beta}(\bar{\gamma}^{\lambda}_{\mu]\gamma}) + 2\bar{\gamma}^{\alpha}_{[\mu|\gamma}\bar{\gamma}^{\lambda}_{|\mu]\alpha} - \bar{\gamma}^{\lambda}_{\alpha\gamma}C^{\alpha}_{\mu\beta} + 2\bar{\gamma}^A_{[\mu|\gamma}\bar{\gamma}^{\lambda}_{|\beta]A} - \bar{\gamma}^{\lambda}_{A\gamma}C^A_{\mu\beta}. \quad (29)$$

The fact that the distribution $D(\Pi)$ is involutive implies that $C^A_{\mu\beta} = 0$. Also if we use here (17a) and the property $\bar{\gamma}^A_{\alpha\beta} = 0$ discussed above we can reduce the previous equation to

$$\bar{R}_{\mu\beta\gamma}{}^{\lambda} = 2\bar{e}_{[\beta}(\gamma^{\lambda}_{\mu]\gamma}) + 2\gamma^{\alpha}_{[\mu|\gamma}\gamma^{\lambda}_{|\mu]\alpha} + \gamma^{\lambda}_{\alpha\gamma}C^{\alpha}_{\mu\beta} = R_{\mu\beta\gamma}{}^{\lambda}. \quad (30)$$

Note that the scalars $R_{\mu\beta\gamma}{}^{\lambda}$ can be understood as the components of the Riemann tensor of the induced metric (first fundamental form) on the leaves of the foliation. Regarding the non-vanishing components of $L^{\Pi\parallel}$ a direct computation using (23) shows

$$L^{\Pi}_{\beta\gamma} = 2\bar{R}_{\beta\alpha\gamma}{}^{\alpha} - \frac{2}{N-p}\bar{R}_{\beta\gamma\alpha}{}^{\alpha} + \frac{\bar{R}^{\Pi}}{1-N+p}\Pi_{\beta\gamma}, \quad \bar{R}^{\Pi} = \bar{R}_{\beta\alpha\gamma}{}^{\alpha}\Pi^{\beta\gamma}, \quad (31)$$

which on using (30) yields

$$L^{\Pi}_{\beta\gamma} = 2R_{\beta\gamma} + \frac{\Pi_{\beta\gamma}R}{1-N+p}, \quad R_{\beta\gamma} \equiv R_{\beta\delta\gamma}{}^{\delta}, \quad R \equiv R_{\beta\gamma}\Pi^{\beta\gamma}, \quad (32)$$

where, for the same reasons pointed out above with the Riemann tensor, $R_{\beta\gamma}$ coincides with the Ricci tensor computed from the first fundamental form and R is its scalar curvature. Combining (30) and (32) with (21)-(22) we obtain after some algebra

$$(T^{\parallel}(\Pi))_{\beta\alpha\gamma}{}^{\delta} = 2R_{\beta\alpha\gamma}{}^{\delta} - \frac{2}{2-N-p}(\Pi^{\delta}_{[\beta}L^{\Pi}_{\alpha]\gamma} + \Pi_{\gamma[\alpha}L^{\Pi}_{\beta]\mu}\Pi^{\mu\delta}) = 2W_{\gamma\alpha\beta}{}^{\delta}, \quad (33)$$

where $W_{\gamma\alpha\beta}{}^{\delta}$ is the Weyl tensor defined by the leaves first fundamental form. Hence, when $N-p > 3$ the tensor $T^{\parallel}(\Pi)$ vanish if and only if these leaves are conformally flat. In a similar fashion as above, from (26) we deduce the only non-vanishing components of $B^{\parallel}(\mathbf{P})$ are $B^{\parallel}(\mathbf{P})_{\alpha\beta\gamma}$ and these are given by

$$B^{\parallel}(\mathbf{P})_{\alpha\beta\gamma} = \bar{\nabla}_{[\gamma}L^{\Pi}_{\beta]\mu} = \nabla_{[\gamma}L^{\Pi}_{\beta]\mu}, \quad (34)$$

where in the last step we used again (17a) and the property $\bar{\gamma}^A_{\alpha\beta} = 0$ in the expansion of $\bar{\nabla}_{[\gamma}L^{\Pi}_{\beta]\mu}$ in terms of the components of the bi-conformal connection. Since $L^{\Pi}_{\beta\mu}$ is, up to a factor, the Schouten tensor of the induced metric on the leaves we conclude that $\nabla_{[\gamma}L^{\Pi}_{\beta]\mu}$ is the Cotton tensor of the same metric and therefore $B^{\parallel}(\mathbf{P})$ will be zero if and only if the Cotton tensor of these leaves vanish. In the case of the leaves being of dimension 3 ($N-p=3$) this is equivalent to their being conformally flat. \square

Corollary 1. *Under the hypotheses of theorem 1 the tensor $\bar{R}^{\parallel}(\mathbf{P})$ is zero if and only if the leaves of the foliation generated by the pair (\mathbf{P}, Π) are flat.*

Proof. See considerations coming after eq. (30). \square

Remark 1. If $N \leq 3$ then the leaves of any foliation are of dimension 1 or 2 and hence they are trivially conformally flat. Therefore to deal with non-trivial situations we need to consider the case $N > 3$ which is the situation assumed by the hypotheses of Theorem 1.

5 Applications

The obvious application of our result comes in the determination of the conformal flatness of the leaves of a foliation. Of course it is fairly trivial to determine this if the foliation is given in an *explicit form* (for example in terms of the level set functions) because in this case it is possible to compute *explicitly* the first fundamental form for each leaf and then one just needs to compute the Weyl or the Cotton tensor for it. However, there are situations where one does not have an explicit form of the foliation or the first fundamental form of the leaves is very difficult to compute. For example, this situation arises if one is interested in investigating the existence of a foliation of a pseudo-Riemannian manifold by conformally flat leaves. To understand better this problem let us consider the special simpler case where the leaves of the foliation have co-dimension $p = 1$. In this case the distribution $D(\mathbf{\Pi})$ can be characterized by a 1-form $\omega \in \Lambda^1(V)$ such that $\omega_x(\vec{X}_x) = 0$, for any $\vec{X}_x \in D_x(\mathbf{\Pi})$, and $x \in V$. In addition, if the distribution $D(\mathbf{\Pi})$ is involutive then, $\omega \wedge d\omega = 0$ (integrable 1-form). Indeed it is possible to establish a relation between ω and the projectors $P, \mathbf{\Pi}$.

Proposition 1. *Any non-degenerate foliation of a pseudo-Riemannian manifold (V, g) with co-dimension 1 leaves can be characterized by orthogonal projectors $P, \mathbf{\Pi}$ adopting the form*

$$P = \frac{\omega^\sharp \otimes \omega}{g^\sharp(\omega, \omega)}, \quad \mathbf{\Pi} = \mathbb{1} - P, \quad (35)$$

where $\omega \in \Lambda^1(V)$ is such that $\omega \wedge d\omega = 0$ and ω^\sharp is the vector determined by the relation $\omega^\sharp(\cdot) = g^\sharp(\omega, \cdot)$ $g^\sharp(\omega, \omega) \neq 0$.

Proof. For any $\omega \in \Lambda^1(V)$ the quantities $\mathbf{\Pi}$ and P introduced by (35), define the distribution

$$D(P) = \text{Span}\{\omega^\sharp\}, \quad (36)$$

and its orthogonal complement $D(\mathbf{\Pi})$. Also, by construction $\omega(\vec{X}) = 0$, for any $\vec{X}_x \in D_x(\mathbf{\Pi})$. If in addition $\omega \wedge d\omega = 0$, then $D(\mathbf{\Pi})$ is an involutive distribution and by Frobenius theorem it generates a foliation of V with co-dimension 1 leaves. Reciprocally, any foliation of V with co-dimension 1 leaves determines an integrable 1-form ω , as explained above. We can then take then equation (35) as the definition for P and $\mathbf{\Pi}$. \square

Now if we wish to adapt the result of Theorem 1 to the case of a foliation with leaves of co-dimension 1 then it is clear that one can write condition (20) or (25) as a condition about the existence of a certain 1-form ω . This condition will be generically not satisfied by any ω unless the pseudo-Riemannian manifold admits a foliation by co-dimension 1 hypersurfaces. See example 1 for the study of a particular case of this situation. Another application arises if we are given an integrable non degenerate 1-form and we are asked whether the foliation it generates is formed by conformally flat leaves. In this case we do not have an explicit form for the leaves, nor their first fundamental form and hence the *classical procedure* cannot be applied directly. This problem does not arise if we apply Theorem 1 as the explicit form of the leaves is not required. With just the knowledge of the integrable 1-form we can build the projectors $P, \mathbf{\Pi}$ via eqs. (35) and then check whether (20) or (25) are fulfilled.

Example 1. Take as pseudo-Riemannian manifold (V, g) the region of the Schwarzschild solution of the vacuum Einstein equations covered by the standard Schwarzschild coordinates (t, r, θ, ϕ)

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \frac{1}{1 - \frac{2M}{r}} dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi), \quad (37)$$

where M is the mass parameter. Suppose we ask ourselves about the existence of a foliation of V whose leaves are conformally flat and given by the family of hypersurfaces

$$\{\Phi(t, r) = c\}, \quad c \in \mathbb{R}, \Phi \in C^2(V).$$

This means that the level set function for such a foliation must take the form $\Phi(t, \phi)$ and hence we can choose the 1-form

$$\omega = d\Phi = \Phi_t dt + \Phi_\phi d\phi, \quad \Phi_t \equiv \frac{\partial \Phi}{\partial t}, \quad \Phi_\phi \equiv \frac{\partial \Phi}{\partial \phi}$$

to construct the pair $(\mathbf{P}, \mathbf{\Pi})$ through eq. (35). Since $N = \dim(V) = 4$ we need to check the condition $\mathbf{B}^{\parallel}(\mathbf{P}) = 0$ in accordance with Theorem 1. Once the pair $(\mathbf{P}, \mathbf{\Pi})$ is given this is an algorithmic computation which can be carried out with the system [5]. The result is an overdetermined system of partial differential equations for the function Φ which we need to study. One of these equations is

$$(B^{\parallel})_{\phi\theta\phi} = \frac{4r^5(2M-r)\Phi_{\phi}^2\Phi_t^4\left(M(9M-4r)\Phi_{\phi}^2+r^4\Phi_t^2\right)\sin^5\theta\cos\theta}{\left((2M-r)\Phi_{\phi}^2+r^3\Phi_t^2\sin^2\theta\right)^4} = 0 ,$$

which is only true if the numerator of the rational expression vanishes. As Φ is a function of just t and ϕ the only possibilities are that either $\Phi_{\phi} = 0$ or $\Phi_t = 0$. The latter possibility can be discarded as one can explicitly check that there are components of $\mathbf{B}^{\parallel}(\mathbf{P})$ which cannot vanish, so we are left with the case $\Phi_{\phi} = 0$ which makes $\mathbf{B}^{\parallel}(\mathbf{P})$ vanish identically. This corresponds to the trivial case of the leaves being the standard static hypersurfaces which, as is well-known, are conformally flat for the Schwarzschild solution.

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